Math 270 Day 5 Part 2

Section 2.4: Exact Equations

Section 2.4: Exact Equations

What we'll go over in this section

- Preliminary Discussion
- What is an exact equation?
- Solving exact equations

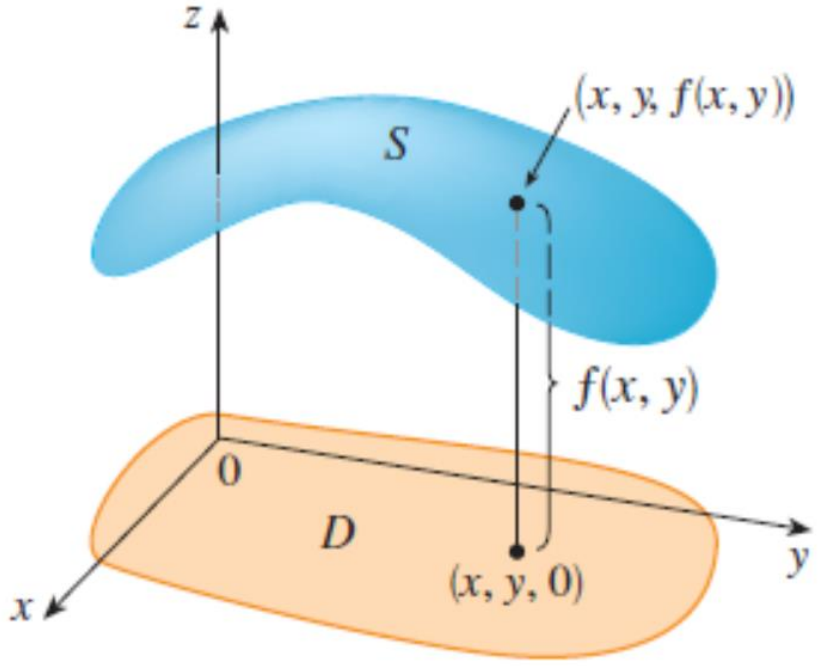
Section 2.4: Exact Equations

Preliminary Discussion

- Graphs of 2 variable functions

Definition

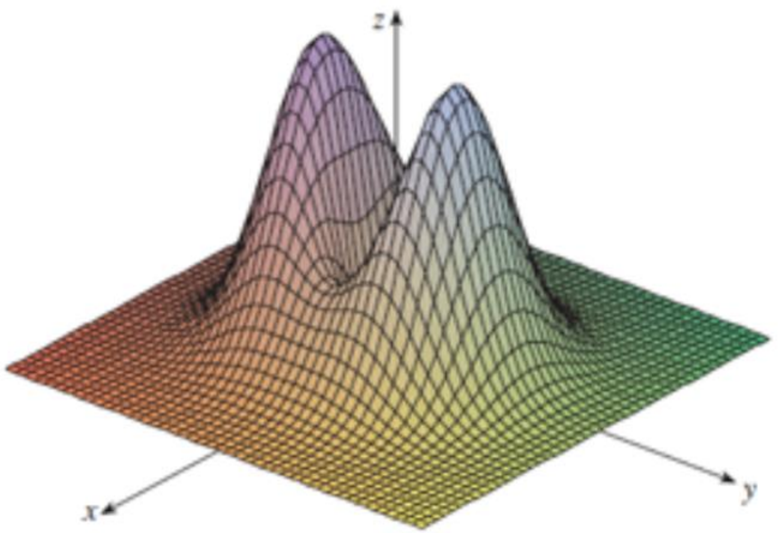
If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

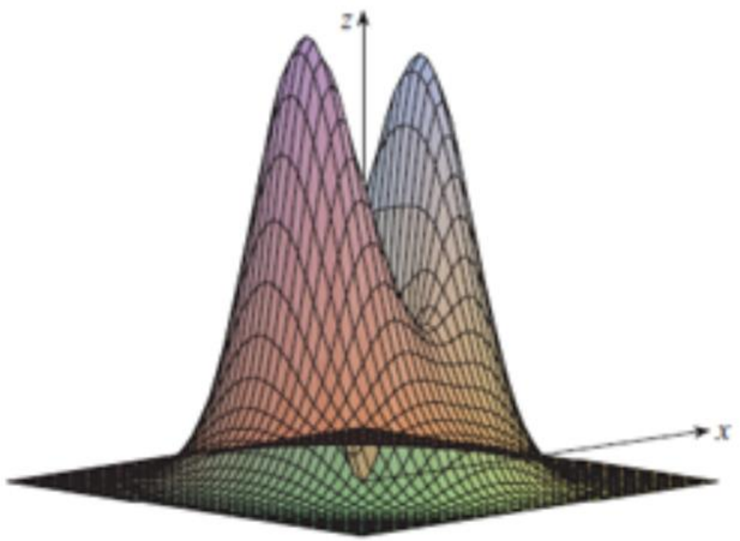


Section 2.4: Exact Equations

Preliminary Discussion

- Graphs of 2 variable functions





$$f(x,y) = \left(x^2 + 3y^2\right)e^{-x^2-y^2}$$

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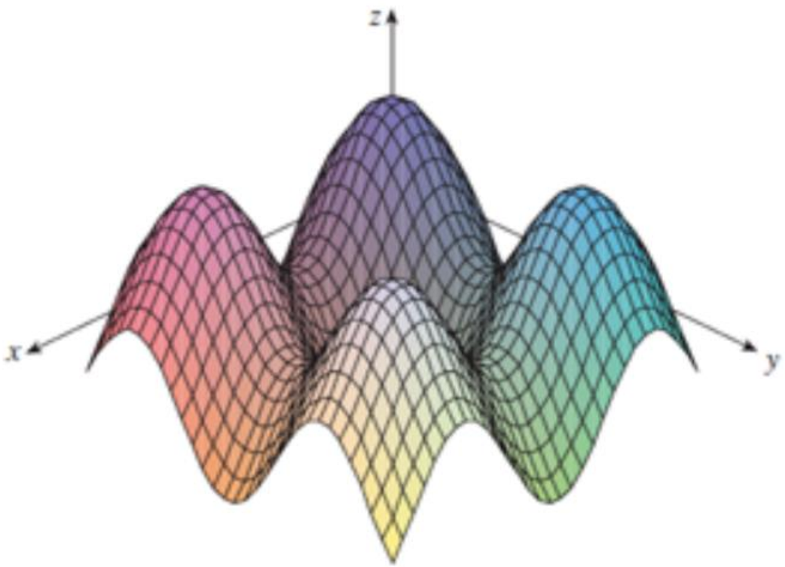

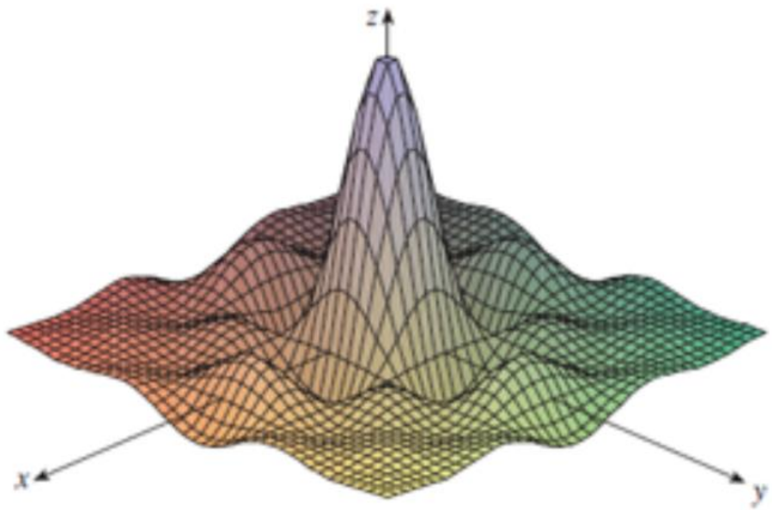

$$f(x,y) = \sin x + \sin y \qquad f(x,y) = \frac{\sin x \sin y}{xy}$$

Section 2.4: Exact Equations

Preliminary Discussion

- Graphs of 2 variable functions
- Level Curves

Definition

The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

Level curves are just traces parallel to the *xy*-plane, projected onto the *xy*-plane.

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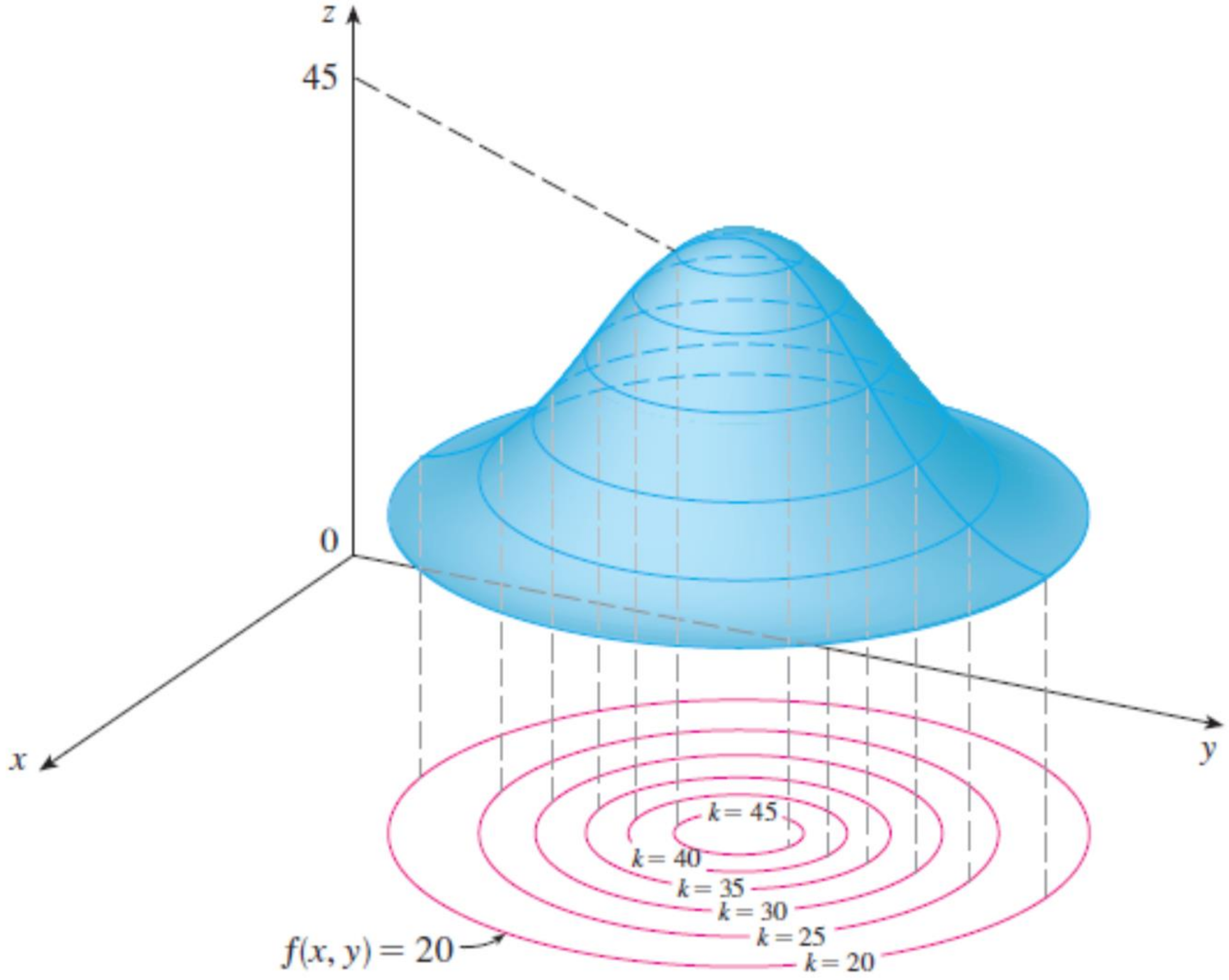
Preliminary Discussion

- Graphs of 2 variable functions
- Level Curves



Section 2.4: Exact Equations

Preliminary Discussion

- Graphs of 2 variable functions
- Level Curves
- All implicit Solutions to DEs are level curves to a 2 variable function
- Chain rule to get dy/dx. This is a DE
- Total differential dF
- Every level curve is a solution to the exact equation
- F(x,y)=C are solutions to $\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$

Section 2.4: Exact Equations

Preliminary Discussion

Example 1 Solve the differential equation

$$\frac{dy}{dx} = -\frac{2xy^2+1}{2x^2y}.$$

Solution Some of the choices of differential forms corresponding to this equation are

$$(2xy^2+1)\,dx + 2x^2y\,dy = 0,$$

$$\frac{2xy^2+1}{2x^2y}dx + dy = 0,$$

$$dx + \frac{2x^2y}{2xy^2+1}dy = 0, \quad \text{etc.}$$

However, the first form is best for our purposes because it is a total differential of the function $F(x, y) = x^2y^2 + x$:

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What is an Exact Equations?

Exact Differential Form

Definition 2. The differential form $M(x, y)\,dx + N(x, y)\,dy$ is said to be **exact** in a rectangle R if there is a function $F(x, y)$ such that

$$\text{(4)} \qquad \frac{\partial F}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y) = N(x, y)$$

for all (x, y) in R. That is, the total differential of $F(x, y)$ satisfies

$$dF(x, y) = M(x, y)\,dx + N(x, y)\,dy\,.$$

If $M(x, y)\,dx + N(x, y)\,dy$ is an exact differential form, then the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is called an **exact equation.**

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What is an Exact Equations?

Test for Exactness

Theorem 2. Suppose the first partial derivatives of $M(x, y)$ and $N(x, y)$ are continuous in a rectangle R. Then

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is an exact equation in R if and only if the compatibility condition

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y) \qquad (5)$$

holds for all (x, y) in R.[†]

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Solving Exact Equations

Method for Solving Exact Equations

(a) If $M\,dx + N\,dy = 0$ is exact, then $\partial F/\partial x = M$. Integrate this last equation with respect to x to get

$$F(x, y) = \int M(x, y)\,dx + g(y)\,. \tag{11}$$

(b) To determine $g(y)$, take the partial derivative with respect to y of both sides of equation (11) and substitute N for $\partial F/\partial y$. We can now solve for $g'(y)$.

(c) Integrate $g'(y)$ to obtain $g(y)$ up to a numerical constant. Substituting $g(y)$ into equation (11) gives $F(x, y)$.

(d) The solution to $M\,dx + N\,dy = 0$ is given implicitly by

$$F(x, y) = C\,.$$

(Alternatively, starting with $\partial F/\partial y = N$, the implicit solution can be found by first integrating with respect to y; see Example 3.)

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Solving Exact Equations

Example 2 Solve $(2xy - \sec^2 x)\,dx + (x^2 + 2y)\,dy = 0$.

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Solving Exact Equations

Example 3 Solve $(1 + e^x y + x e^x y)\,dx + (x e^x + 2)\,dy = 0$.

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Solving Exact Equations

Example 4 Show that

$$(x + 3x^3 \sin y)\,dx + (x^4 \cos y)\,dy = 0 \qquad (16)$$

is *not* exact but that multiplying this equation by the factor x^{-1} yields an exact equation. Use this fact to solve (16).